VIRILE AGITUR

Knox Grammar School

2023

Trial Higher School Certificate Examination

Name: ______________________

Teacher: ____________________

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- The official NESA Reference Sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

Teachers:

Mr Bradford (Examiner)
Ms Yun

Section I ~ Pages 3-8

- 10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II ~ Pages 9-16

- 90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Write your name, your Teacher's Name and your Student Number on the front cover of each answer booklet

Number of Students in Course: 37

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Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1. A ball of mass m kilograms is projected vertically up from ground level.

The forces acting on the ball as it moves up are its weight mg and the air resistance mkv^2 where v m/s is its speed and k is a positive constant.

The distance of the ball from the ground at time t seconds is x metres. The positive direction is taken to be upwards. What is the equation of motion?

(A) $\dfrac{dv}{dt} = g - kv^2$

(B) $\dfrac{dv}{dt} = -m\,g - mkv^2$

(C) $\dfrac{dv}{dt} = -\ g + kv^2$

(D) $\dfrac{dv}{dt} = -g - kv^2$

2. A particle is moving in simple harmonic motion. The velocity v of the particle is given by $v = 1 + 3\cos\left(2t - \frac{\pi}{4}\right)$.

Which of the following is the first time the acceleration of the particle is a maximum?

(A) $\frac{7\pi}{8}$

(B) $\frac{3\pi}{8}$

(C) $\frac{\pi}{8}$

(D) $\frac{\pi}{4}$

3. Consider the vector $\underset{\sim}{u} = 2p\underset{\sim}{i} - p\underset{\sim}{j} + 2p\underset{\sim}{k}$, where p is a positive constant.

Which of the following vectors has the same direction as $\underset{\sim}{u} = 2p\underset{\sim}{i} - p\underset{\sim}{j} + 2p\underset{\sim}{k}$, but for which its magnitude is 3?

(A) $3p\begin{pmatrix} 2p \\ -p \\ 2p \end{pmatrix}$

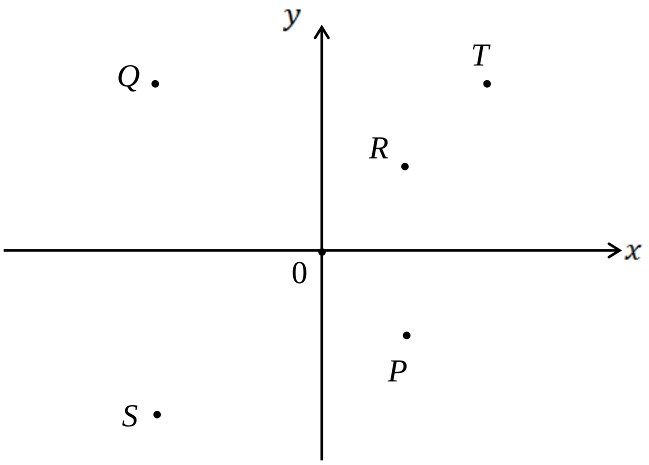
(B) $p\begin{pmatrix} 2p \\ -p \\ 2p \end{pmatrix}$

(C) $\frac{1}{p}\begin{pmatrix} 2p \\ -p \\ 2p \end{pmatrix}$

(D) $\frac{3}{p}\begin{pmatrix} 2p \\ -p \\ 2p \end{pmatrix}$

4. In the Argand diagram the point P represents a complex number.

When this number was multiplied by $\left(\frac{1+i}{2\sqrt{2}}\right)^4 (2i)^5$ it gave a new complex number.



Which of the following points represents the new complex number?

(A) Q

(B) R

(C) T

(D) S

5. Consider the statement: "For all quadrilaterals, if the four angles are right angles, then the quadrilateral is a rectangle."

Which of the following is the contrapositive of the statement?

(A) There exists a quadrilateral such that the four angles are not right angles and the quadrilateral is not a rectangle.

(B) There exists a quadrilateral such that the quadrilateral is not a rectangle and the four angles are not right angles.

(C) For all quadrilaterals, if the quadrilateral is not a rectangle, then the four angles are not right angles.

(D) For all quadrilaterals, if the four angles are right angles, then the quadrilateral is not a rectangle.

6. Consider the statement: " $\forall\, x \in \mathrm{N}$, x is even $\Rightarrow x^2(x^2+4)$ is divisible by 16".

Which of the following is the negation of the statement?

(A) $\exists\, x \in \mathrm{N}$, x is not even and $x^2(x^2+4)$ is not divisible by 16.

(B) $\exists\, x \in \mathrm{N}$, x is even and $x^2(x^2+4)$ is not divisible by 16.

(C) $\forall\, x \in \mathrm{N}$, x is even and $x^2(x^2+4)$ is not divisible by 16.

(D) $\forall\, x \in \mathrm{N}$, x is even or $x^2(x^2+4)$ is not divisible by 16.

7. Given that α is a complex root of the equation $z^3 = -i$ and $\mathrm{Re}\,(\alpha) > 0$, simplify

$$\frac{\alpha^2 - i\overline{\alpha}}{\alpha^5 + 2\overline{\alpha}}.$$

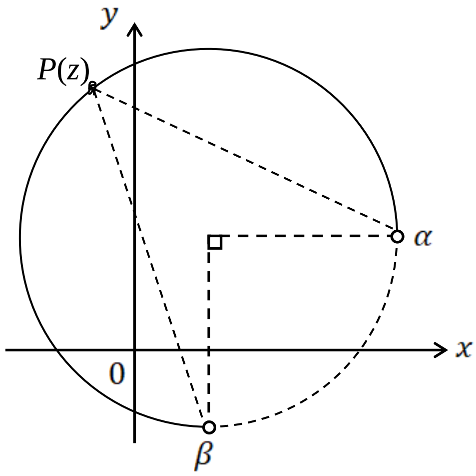
(A) $-2i$

(B) $2\,i$

(C) $-\,i$

(D) i

8. The diagram shows the solution of an equation as traced out by the point $P(z)$. The path traced out by the point P representing the complex number z is a three-quarter circle.



Which of the following could be the equation?

(A) $\text{Arg}\,(z-\alpha)-\text{Arg}\,(z-\beta)=0$

(B) $\text{Arg}\,(z-\alpha)-\text{Arg}\,(z-\beta)=\frac{\pi}{4}$

(C) $\text{Arg}\,(z-\alpha)-\text{Arg}\,(z-\beta)=\frac{\pi}{2}$

(D) $\text{Arg}\,(z-\beta)-\text{Arg}\,(z-\alpha)=\frac{\pi}{4}$

9. Consider the vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB}$ with $|\overrightarrow{OB}| = 3$.

Given that $\overrightarrow{OA}.\overrightarrow{OB} = 6$, find in square units the area of triangle OAB?

(A) $3\sqrt{10}$

(B) $\dfrac{3\sqrt{14}}{2}$

(C) $\dfrac{3\sqrt{10}}{4}$

(D) $\dfrac{3\sqrt{10}}{2}$

10. Consider the complex number $\alpha = \left(1 + i\sqrt{3}\right)\left(\sqrt{3} + i\right)^p$, where p is an integer.

What are the values of p for which α is purely real?

(A) $p = -2 + 6n$, where n is an integer.

(B) $p = 2 + 6n$, where n is an integer.

(C) $p = -2 + 3n$, where n is an integer.

(D) $p = -2 + 12n$, where n is an integer.

End of Section I

Section II

90 marks
Attempt questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a separate writing booklet.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the complex numbers $\alpha = 2\, e^{i\frac{\pi}{6}}$ and $\beta = 2\, e^{i\frac{\pi}{3}}$. **2**

Find the value of $\alpha^3 + \beta^3$, giving your answer in the form $a + ib$.

(b) (i) Prove $n^3 + 5n$ is divisible by 6 for all integers $n \geq 1$using mathematical induction. **4**

(ii) Consider the statement $P(n)$ in the set of integers. **2**

"If $2n^3 + 40n$ is multiple of 96, then n is even."

Prove that the converse of $P(n)$ is true.

(c) Find $\displaystyle\int \frac{\sin 2x + \sin x}{2\cos x + 1}\, dx$. **2**

(d) Shade the region on the Argand diagram where the two inequalities **2**
$|\, z - 2 + i \,| \leq 2$ and $Im\, z \leq 0$ both hold.

(e) Consider the two vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \alpha\underset{\sim}{i} + \beta\underset{\sim}{k}$. **3**

If the angle between the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{\pi}{4}$ and $|\underset{\sim}{b}| = \sqrt{2}$, find the possible values of α and β.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle starts at the origin with velocity $v = 3$ and acceleration given by $a = v\sqrt{36 - v^2}$, where v is the velocity of the particle. **3**

Find an expression for the velocity v in terms of the displacement x.

(b) Two lines, L_1 and L_2, are represented respectively by the following vector equations:

$$\underset{\sim}{r}_1 = \begin{pmatrix} p \\ -2 \\ q \end{pmatrix} + \lambda_1 \begin{pmatrix} -6 \\ 2 \\ -5 \end{pmatrix} \text{ and } \underset{\sim}{r}_2 = \begin{pmatrix} -2 \\ -8 \\ -4 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \text{ where } \lambda_1 \text{ and } \lambda_2 \in \mathbb{R}.$$

The two lines intersect at the point $A(-2r, r, -4r)$.

(i) Find the value of r and hence write down the coordinates of A. **2**

(ii) Hence, find p and q. **2**

(c) Solve $iz^2 - (i - 2)z + 1 + 3i = 0$. Give your answers in the form $a + bi$, where a and b are real. **4**

(d) Evaluate $\displaystyle\int_0^{\pi/2} \frac{3}{8\cos x + 10}\, dx$. **4**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The complex number $z = -2 + i$ is a root of $z^3 + 2z^2 + pz + q = 0$, where p and q are real numbers. **3**

What are the values of p and q?

(b) The sphere H has its centre $A\,(-2, -1, 1)$ and passes through $B(\,1,\ 5,\ 3\,)$.

(i) Find the cartesian equation of H. **2**

(ii) The intersection of sphere H and a second sphere K with equation $(x+2\,)^2 + (y-3\,)^2 + (z-1\,)^2 = 25$ is a circle. **3**

Find the equation of the circle.

(c) (i) Show that $\cos(\,\alpha+\beta\,) + \cos\,(\,\alpha-\beta\,) = 2\cos\alpha\ \cos\beta$. **1**

(ii) Hence, or otherwise, find all the solutions of: **3**

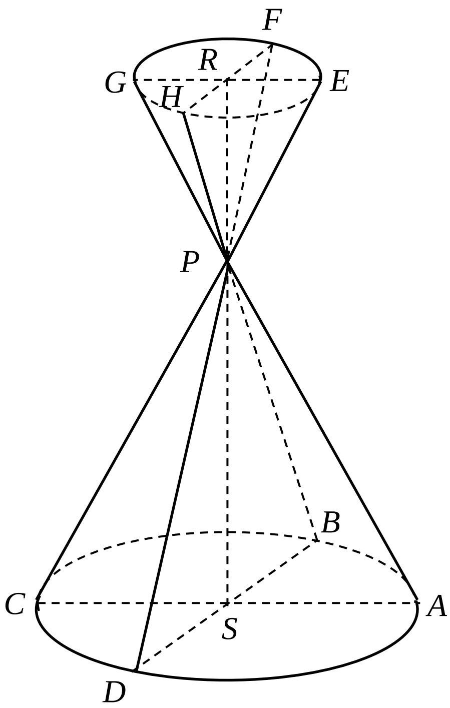
$$\cos\frac{3x}{4} + 2\cos\frac{x}{4}\sin x + \cos\frac{x}{4} = 0 \text{ for } 0 \le x \le 2\pi.$$

Question 13 continues on page 12

Question 13 (continued)

(d) The diagram shows a small cone on the top of a larger cone. The two cones have a common axis of symmetry passing by the centres R and S of their circular bases and their common apex P. **3**

The top cone has a radius r and height h and the bottom cone has radius $2r$ and height $2h$.



Show that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PG} + \overrightarrow{PH} = 2\,\overrightarrow{PS}$.

End of Question 13

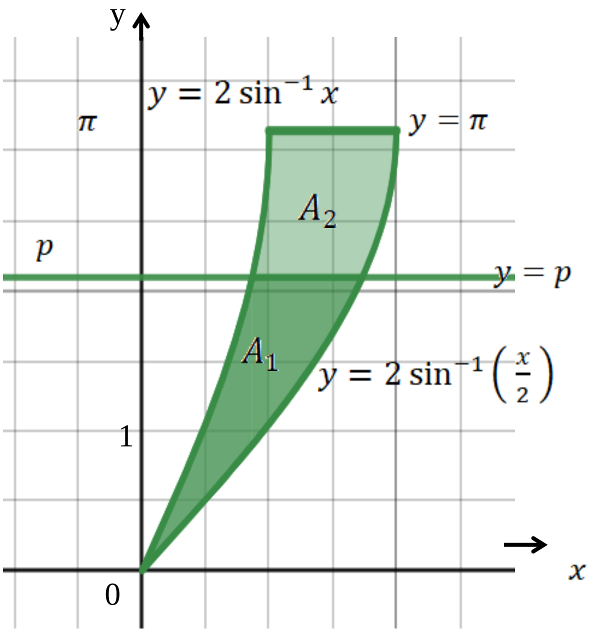
Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle is moving along the x axis in simple harmonic motion centred at $x = 2$ and with amplitude 5. **3**

The velocity of the particle is given by $v^2 = p + 8x - qx^2$, where p and q are constants.

Find the exact numerical value for the period of the motion.

(b) The shaded region is enclosed by the curves $y = 2\sin^{-1} x$, $y = 2\sin^{-1}\left(\frac{x}{2}\right)$, and the line $y = \pi$ in the first quadrant. It is divided into two parts A_1 and A_2 by the line $y = p$ where $\frac{\pi}{2} < p < \pi$.



The volumes of the solids of revolution formed when the two areas A_1 and A_2 are rotated about the y axis are respectively V_1 and V_2 and $V_1 = 2V_2$.

Given that $\sin p \approx p^2 - 2p - 1.06612$, find the value of p correct to 5 decimal places. **4**

Question 14 continues on page 13

Question 14 (continued)

(c) Prove by contradiction that $\sqrt[3]{3}$ is irrational. **3**

(d) Provide a *non-inductive* proof as to why $12^n > 5^n + 7^n$ for all integers $n \geq 2$. **2**

(e) Prove the following logical equivalence for *n* a natural number: **3**
"$n + 6$ is a positive odd integer if and only if $n + 21$ is a positive even integer".
Explain your reasoning carefully.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) It is given that $3^x > x + \ln(x)$ for $x > 0$. By forming a series of n inequalities, show that $3^{n+1} > n^2 + n + 3 + 2\ln(n!)$, where n is a positive integer. Do **NOT** use induction. **4**

(b) Two particles P_1 and P_2 start moving from a point A at the same time, along different straight lines.

The positions of P_1 and P_2 at a time t are $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ respectively as shown below, where $a > 1$.

$$\underset{\sim}{r_1} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} \ln a \\ \ln a \\ 2 \end{pmatrix} \text{ and } \underset{\sim}{r_2} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 + \ln a \\ 2 + 2\ln a \\ 3\ln a \end{pmatrix}$$

After one second the distance between the two particles is 6 m.

(i) Show that $\ln a = 2$. **2**

(ii) What is distance between the two particles after 2 seconds? **2**

(c) A particle P of mass m kg is projected vertically upwards from the origin O with initial velocity v_o m s^{-1} in a certain medium. **3**

The particle is subject to both a constant gravitational force mg and a resistant force $m\sqrt{\dfrac{v}{g}}$ where v m s^{-1} is the velocity of the particle at time t after being projected and g is the acceleration due to gravity.

The equation of motion is $m\dfrac{dv}{dt} = -mg - m\sqrt{\dfrac{v}{g}}$.

(i) Show that $t = -\int \dfrac{g}{g^2 + \sqrt{gv}}\, dv$. **1**

(ii) By using the substitution $u = \sqrt{gv}$ in the above integral, show that t and v are related by the equation: **3**

$$t = 2\left(\sqrt{gv_o} - \sqrt{gv}\right) + 2g^2 \ln\left(\frac{g^2 + \sqrt{gv}}{g^2 + \sqrt{gv_o}}\right)$$

(iii) Hence, find a completely simplified algebraic expression for how long it takes the particle to drop its velocity from $\dfrac{v_o}{9}$ to $\dfrac{v_o}{81}$. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove that the sum of the squares of three consecutive integers that are even and positive, always gives a remainder 8 when divided by 12. **2**

(b) A particle is initially at $x = \frac{1}{2}$ and has an initial velocity $1\ ms^{-1}$.

The acceleration of the particle is given by $a = \frac{-1}{2x^2}$, where x metres is the displacement of the particle from the origin.

(i) Prove that $\frac{dx}{dt} = \sqrt{\frac{1-x}{x}}$ **3**

(ii) Using the substitution $x = \sin^2\theta$, show that the time T taken for the particle to reach $x = 1$ is $T = \left(\frac{\pi}{4} + \frac{1}{2}\right)$ seconds. **4**

(c) Let $I_n = \int_0^1 \frac{x^3}{(2 - x^2)^n}\, dx$ where $n \geq 1$.

(i) Show that $I_{n+1} = \frac{1}{4n} + \frac{n-2}{2n} I_n$ **4**

You may use $\frac{x^5}{(2-x^2)^{n+1}} = \frac{A\,x^3}{(2 - x^2)^n} + \frac{B\,x^3}{(2 - x^2)^{n+1}}$, where A and B are constants to be determined.

(ii) Hence, or otherwise, find the exact value of $\int_0^1 \frac{x^3}{(2 - x^2)^2}\, dx$ **2**

End of Paper

2023 Year 12 Mathematics Extension 2 Task 4 Trial -Worked Solutions

The following solutions are typical of what a competent Extension 2 student would demonstrate in an assessment setting. They are not meant to be exhaustive nor do they reflect 'best practice'. The algebra is deliberately laboured to accommodate the needs of all candidates; at least that is the goal.

1. **D**

We take the positive direction as upwards. The forces acting on the ball are gravity $\overrightarrow{mg}$ which is acting downwards which means it is negative and also air resistance of $\overrightarrow{mkv^2}$ which is against the direction of the motion, which is also negative.
Projecting these forces vertically, we get

$ma = -mg - mkv^2$ but $a = \dfrac{dv}{dt}$ - This means

$m\frac{dv}{dt} = -mg - mkv^2$ that is $\frac{dv}{dt} = -g - kv^2$

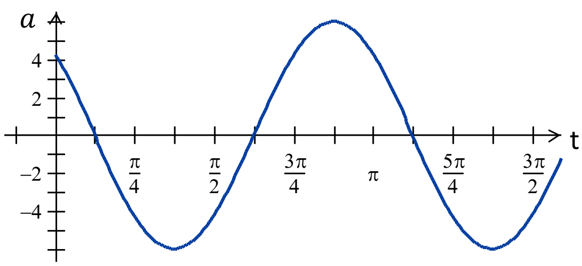
Hence, the correct option is **D**.

2. **A**

As $v = 1 + 3\,cos\left(2t - \frac{\pi}{4}\right)$ then

$a = -6\,sin\left(2t - \frac{\pi}{4}\right)$

The graph of acceleration versus time is shown below.



The maximum value of the acceleration is 6, so to find the time the acceleration of the particle is a maximum, we let $a = 6. We\ get$

$-6\,sin\left(2t - \dfrac{\pi}{4}\right) = 6$

$sin\left(2t - \dfrac{\pi}{4}\right) = -1 = \sin\left(-\dfrac{\pi}{2}\right)$

$2t - \frac{\pi}{4} = -\frac{\pi}{2} + 2n\,\pi$ or $2t - \frac{\pi}{4} = \pi + \frac{\pi}{2} + 2n\,\pi$

$2t = -\dfrac{\pi}{4} + 2n\,\pi \qquad 2t = \dfrac{7\pi}{4} + 2n\,\pi$

$t = -\dfrac{\pi}{8} + n\,\pi \qquad t = \dfrac{7\pi}{8} + n\,\pi$

When $n = 1, t = \frac{7\pi}{8}$ is the smallest positive value for t for which the acceleration is a maximum.

Consequently, $t = \frac{7\pi}{8}$ is the first time the acceleration of the particle is a maximum.
Hence, the correct option is **A**.

3. **C**

The vector that has the same direction as $\underset{\sim}{u}$, and so it must be of the form

$k\begin{pmatrix} 2p \\ -p \\ 2p \end{pmatrix}$, where k is constant real number.

The length of this vector is

$k\sqrt{(2p)^2 + (-p)^2 + (2p)^2}$

$= k\sqrt{4p^2 + p^2 + 4p^2}$

$= 3kp$ as p is a positive constant.

As the length of this vector is 3 units then $3kp = 3$

that is $k = \frac{1}{p}$. Therefore, the required vector is $\frac{1}{p}\begin{pmatrix} 2p \\ -p \\ 2p \end{pmatrix}$

Hence, the correct option is **C**.

4. **D**

$1 + i = \sqrt{2}\,cis\,\frac{\pi}{4}$ so $\left(\dfrac{1\,+\,i}{2\sqrt{2}}\right)^4 = \left(\dfrac{\sqrt{2}\,cis\,\frac{\pi}{4}}{2\sqrt{2}}\right)^4$

$= \left(\dfrac{1}{2}\right)^4 cis\,\dfrac{4\,\pi}{4} = \dfrac{1}{16}cis\,\pi$

$(2i\,)^5 = 32\,i^5 = 32\,i = 32\,cis\left(\frac{\pi}{2}\right)$

Hence, $\left(\dfrac{1\,+\,i}{2\sqrt{2}}\right)^4 (2i\,)^5 = \frac{1}{16}\,cis\,\pi \times 32\,cis\left(\dfrac{\pi}{2}\right)$

$= 2\,cis\left(\frac{3\pi}{2}\right)$

So, the new complex number will be twice as far from the origin as P. Also, the new complex number will have an argument which is rotated $\frac{3\pi}{2}$ $radians$ anticlockwise from O. This indicates that the new complex number will be at point S.
Hence, the correct option is **D**.

5. **C**

For a statement "For all, if P then Q ", its contrapositive statement is "For all, if not Q then not P ".

So, as the statement is "For all quadrilaterals, if the four angles are right angles, then the quadrilateral is a rectangle", the contrapositive statement would be "For all quadrilaterals, if the quadrilateral is not a rectangle, then the four angles are not right angles."

Hence, the correct option is **C**.

6. **B**

Consider the statement " For all values of x, which are elements of the Natural numbers, if x is even then $x^2(x^2+4)$ is divisible by 16."

The negation statement would be "There exists a value of x, which is an element of the Natural numbers, such that if x is even then $x^2(x^2+4)$ is not divisible by 16."

Hence, the correct option is **B**.

7. A

Let $z = rcis\,\theta$ this means $z^3 = r^3$and$-i = cis\ -\frac{\pi}{2}$

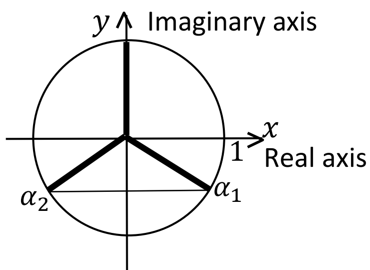
So, $z^3 = -i$ can be written as $r^3 cis\,3\theta = cis\,(-\frac{\pi}{2} + 2k\pi)$.

This indicates that $r^3 = 1$ which means $r = 1$ as $r > 0$ and $cos\,3\theta = cos\left(-\frac{\pi}{2} + 2k\pi\right) with\ k \in \mathbb{Z}$.

So, $3\theta = -\frac{\pi}{2} + 2k\pi\ \ that\ is\ \ \theta = -\frac{\pi}{6}\,,\ \frac{\pi}{2}\ or -\frac{5\pi}{6}$.

Therefore, the three roots of $z^3 = -i$ are $cis\left(-\frac{\pi}{6}\right)$, $cis\left(\frac{\pi}{2}\right)\ or\ cis\left(-\frac{5\pi}{6}\right)$.

The diagram below shows the three roots of the equation $z^3 = -i$. Of these three roots $\alpha_1 = cis\left(-\frac{\pi}{6}\right)$ and this is the only root with Re$(\alpha) > 0$.



So, $\alpha = cis\left(-\frac{\pi}{6}\right)$

Therefore, $\alpha^2 = cis\left(-\frac{\pi}{3}\right)$

$$= cos\left(-\frac{\pi}{3}\right) + i\,sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i.$$

Also, $i\,\bar{\alpha} = cis\left(\frac{\pi}{2}\right) \times cis\left(\frac{\pi}{6}\right) = cis\left(\frac{2\pi}{3}\right)$

$$= cos\left(\frac{2\pi}{3}\right) + i\,sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i.$$

From the above, $\alpha^2 - i\,\bar{\alpha} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)$

$$= 1 - \sqrt{3}\,i.$$

Now, $\alpha^5 = cis\left(-\frac{5\pi}{6}\right) = cos\left(-\frac{5\pi}{6}\right) + i\,sin\left(-\frac{5\pi}{6}\right)$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Also, $2\bar{\alpha} = 2\,cis\left(\frac{\pi}{6}\right) = 2\left(cos\left(\frac{\pi}{6}\right) + i\,sin\left(\frac{\pi}{6}\right)\right)$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i.$$

So, $\alpha^5 + 2\,\bar{\alpha} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i + \sqrt{3} + i$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Therefore, $\dfrac{\alpha^2 - i\bar{\alpha}}{\alpha^5 + 2\bar{\alpha}} = \dfrac{1 - \sqrt{3}\,i}{\frac{\sqrt{3}}{2} + \frac{1}{2}i} = \dfrac{2(1 - \sqrt{3}\,i)}{\sqrt{3} + i}$

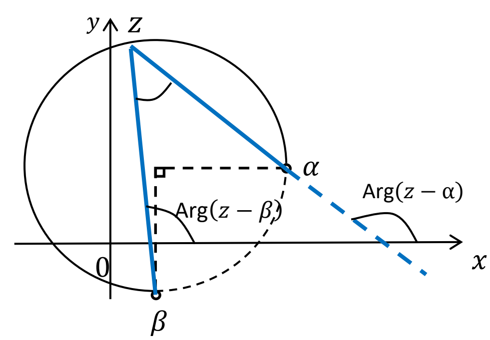
$$= \frac{2(1 - \sqrt{3}\,i)}{\sqrt{3} + i} \times \frac{\sqrt{3} - i}{\sqrt{3} - i}$$

$$= \frac{2(\sqrt{3} - i - 3i - \sqrt{3})}{3 + 1} = \frac{2 \times -4i}{4}$$

$$= -2i$$

Hence, the correct option is **A**.

8. **B**



z lies on the circumference of a circle, radii from α and β meet at the centre at $\frac{\pi}{2}$ $radians$. This means the angle at z must be $\frac{\pi}{4}$ $radi$ans as the angle at the circumference is half the angle at the centre when subtended by the same arc.

OR

$z - \alpha$ represents the vector from α to z.

$z - \beta$ represents the vector from β to z.

Using the exterior angle of a triangle, we can see that the angle at $z = Arg(z - \alpha) - Arg(z - \beta)$

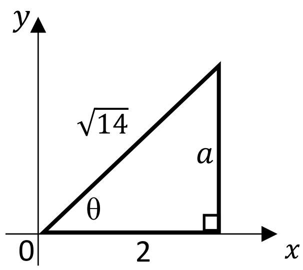
$So,\ Arg(z - \alpha) - Arg(z - \beta) = \frac{\pi}{4}$

Hence, the correct option is **B**.

9. D

$|\overrightarrow{OA}| = \sqrt{4+1+9} = \sqrt{14}$

As $\overrightarrow{OA}.\overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}|cos\theta$ then $6 = \sqrt{14} \times 3 \times cos\theta$; that is, $\cos\theta = \frac{2}{\sqrt{14}}$. As $\overrightarrow{OA}.\overrightarrow{OB} > 0$ and as θ is an angle in a triangle, then θ is acute. By constructing a right angle triangle in the first quadrant and using Pythagoras theorem, we get



$a = \sqrt{(\sqrt{14})^2 - 2^2} = \sqrt{10}.$

Hence, $\sin\theta = \frac{\sqrt{10}}{\sqrt{14}}$.

$So, A = \frac{1}{2} \times 3 \times \sqrt{14} \times \frac{\sqrt{10}}{\sqrt{14}} = \frac{3\sqrt{10}}{2}$

Hence, the correct option is **D**.

10. A

$1 + i\sqrt{3} = 2\,cis\frac{\pi}{3}$ and

$\sqrt{3} + i = 2\,cis\frac{\pi}{6}$ that is

$(\sqrt{3}+i)^p = 2^p\,cis\frac{p\pi}{6}$

So, $\alpha = (1 + i\sqrt{3})(\sqrt{3}+i)^p$

$= 2\,cis\frac{\pi}{3} \times 2^p\,cis\frac{p\pi}{6}$

$= 2^{p+1}\,cis\left(\frac{\pi}{3} + \frac{p\pi}{6}\right)$

$= 2^{p+1}\left(cos\left(\frac{\pi}{3} + \frac{p\pi}{6}\right) + i\,sin\left(\frac{\pi}{3} + \frac{p\pi}{6}\right)\right)$

α is purely real when $Im(\alpha) = 0$; that is,

$sin\left(\frac{\pi}{3} + \frac{p\pi}{6}\right) = 0$

$sin\left(\frac{\pi}{3} + \frac{p\pi}{6}\right) = \sin n\pi$, with $n \in \mathbb{Z}$.

So $\frac{\pi}{3} + \frac{p\pi}{6} = n\pi$

Multiplying by $\frac{6}{\pi}$ on both sides , we get

$2 + p = 6n$ that is $p = -2 + 6n$.

Hence, the correct option is **A**.

QUESTION 11

a) $\alpha = 2e^{i\frac{\pi}{6}}$ and $\beta = 2e^{i\frac{\pi}{3}}$

$\alpha = 2\,cis\,\frac{\pi}{6}$ and $\beta = 2cis\,\frac{\pi}{3}$

$\alpha^3 = 8\,cis\,\frac{\pi}{2}$ and $\beta^3 = 8\,cis\,\pi$

$\alpha^3 + \beta^3 = 8\left(cis\,\frac{\pi}{2} + cis\,\pi\right)$

$= 8\left(cos\frac{\pi}{2} + i\,sin\frac{\pi}{2} + cos\pi + i\,sin\pi\right)$

$= 8\,(0 + i - 1 + 0) = -8 + 8i$

b) (i) We will use mathematical induction to prove the statement that $n^3 + 5n$ is a multiple of 6.

$For\ n = 1$, we get $1^3 + 5 \times 1 = 6$, which is a multiple of 6. So, the statement is true for $n = 1$.

Assume the statement is true for $n = k$, that is

$k^3 + 5k = 6p$, where p is a positive integer.

For $n = k + 1$, we get

$n^3 + 5n = (k+1)^3 + 5(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 5k + 5$

$= k^3 + 5k + 3k^2 + 3k + 6$

$= 6p + 3k(k+1) + 6$ (from assumption)

$= 6(p+1) + 3k(k+1)$

Now, $k(k+1)$ is the product of two consecutive integers. So, one of them must be even and this indicates that $k(k+1)$ is always even.

Therefore, $k(k+1) = 2r$, where r is a positive integer.

So, $n^3 + 5n = 6(p+1) + 3 \times 2r$

$= 6(p+1) + 6r$

$= 6\,(p + 1 + r) = 6h$, where $h \in \mathbb{N}$.

Therefore, if the statement is true for $n = k$, it is also true for $n = k + 1$.

The statement $n^3 + 5n$ was proven true for $n = 1$ and by mathematical induction it is true for $n = 2, n = 3$ and so on. Hence, it is true for all integer values $n \geq 1$.

b) (ii)

The converse of P would be " If n is even, then $2n^3 + 40n$ will be a multiple of 96 ".

Proof:

As n is even, let $n = 2m$, where m is an integer.

So $2n^3 + 40n = 2(2m)^3 + 40 \times 2m$

$= 2 \times 8m^3 + 80m$

$= 16\,(m^3 + 5m)$

Now, in order to prove $16\,(m^3 + 5m)$ is a multiple of 96 we need to prove that $m^3 + 5m$ is a multiple of 6.

We appeal to the inductive result in (b) (i).

Hence, the converse of P which is " If n is even, then $2n^3 + 40n$ will be a multiple of 96 " is true.

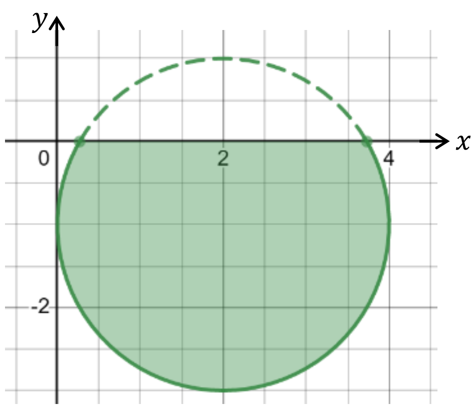
c) $\sin 2x + \sin x = 2\sin x\cos x + \sin x$

$= \sin x\,(2\cos x + 1)$

So, $\int \frac{\sin 2x + \sin x}{2\cos x + 1}\,dx = \int \frac{\sin x\,(2\cos x + 1)}{2\cos x + 1}\,dx$

$= \int \sin x\,dx = -\cos x + c$

d) $|z-2+i| \le 2$ is equivalent to $|z-(2-i)| \le 2$ which describes all values for z, where the length of the vector from $2-i$ to z is less than, or equal to, 2 units long.
This is represented as a circle with radius 2 and centre$(2,-1)$ on the Argand diagram.
$Im(z) \le 0$ represents all points below the Real axis on the Argand diagram. The graph below shows the region where both of these conditions hold simultaneously.



e) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{\alpha}| \times |\underset{\sim}{\beta}| \times \cos\theta$

As $|\underset{\sim}{a}| = \sqrt{4+4+1} = \sqrt{9} = 3$

and $|\underset{\sim}{b}| = \sqrt{2}$ then

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3\sqrt{2}\cos\frac{\pi}{4} = 3\sqrt{2} \times \frac{1}{\sqrt{2}} = 3$

Also, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\alpha + \beta$ and this indicates

$2\alpha + \beta = 3$ (1)

Since. $|\underset{\sim}{b}| = \sqrt{2}$ and $\underset{\sim}{b} = \alpha\, \underset{\sim}{i} + \beta\, \underset{\sim}{k}$ then

$\sqrt{\alpha^2+\beta^2} = \sqrt{2}$. By squaring both sides , we get

$\alpha^2+\beta^2 = 2$ (2)

Now, we need to solve two equations (1) and (2)

simultaneously to find α and β.

From (1), we have $\beta = 3-2\alpha$
By substituting this equation in (2), we get
$\alpha^2 + (3-2\alpha)^2 = 2$
$\alpha^2 + 9 - 12\alpha + 4\alpha^2 = 2$
$5\alpha^2 - 12\alpha + 7 = 0$
$(5\alpha-7)(\alpha-1) = 0$
So, $\alpha = \frac{7}{5}$ *or* $\alpha = 1$.
Now, when
$\alpha = \frac{7}{5}, \beta = 3 - 2 \times \frac{7}{5} = \frac{1}{5}$
Also, $\alpha = 1, \beta = 3 - 2 \times 1 = 1$

QUESTION 12

a) $a = v\frac{dv}{dx} = v\sqrt{36-v^2}$ that is $\frac{dv}{dx} = \sqrt{36-v^2}$

So, $\frac{dx}{dv} = \frac{1}{\sqrt{36-v^2}}$ that is $x = sin^{-1}\left(\frac{v}{6}\right) + c$
When $t=0$, $x=0$ and $v=3$
$0 = sin^{-1}\left(\frac{3}{6}\right) + c$
$0 = \frac{\pi}{6} + c$ that is $c = -\frac{\pi}{6}$
Therefore, $x = sin^{-1}\left(\frac{v}{6}\right) - \frac{\pi}{6}$

that is $x + \frac{\pi}{6} = sin^{-1}\left(\frac{v}{6}\right)$

this indicates $\sin\left(x+\frac{\pi}{6}\right) = \frac{v}{6}$

Hence, $v = 6\,sin\left(x+\frac{\pi}{6}\right)$.

b) i) The line $\underset{\sim}{r_2} = \begin{pmatrix} -2 \\ -8 \\ -4 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ passes

through the point A $(-2r, r, -4r)$ this means
$-2-\lambda_2 = -2r$ (1)
$-8+2\lambda_2 = r$ (2)
$-4-2\lambda_2 = -4r$ (3)
By solving two of these equations simultaneously we can calculate the value of λ_2 at A and the value of r.
Multiplying (1) by 2 , we get $-4-2\lambda_2 = -4r$ (4)
Now, by adding (4) and (2), we get
$-12 = -3r$; that is, $r = 4$.
Also, $-2-\lambda_2 = -8$ that is $\lambda_2 = 6$.
Hence, the point A is $(-8, 4, -16)$.

ii) The line $\underset{\sim}{r_1}$ passes through the point A.
So, by substituting the point coordinates of A we can find the value of λ_1 at A and hence we can the values of p and q.
$p - 6\lambda_1 = -8$ (5)
$-2 + 2\lambda_1 = 4$ (6)
$q - 5\lambda_1 = -16$ (7)
From (6), we have $-2+2\lambda_1 = 4$ that is $2\lambda_1 = 6$ which means $\lambda_1 = 3$.
From (5),we have $p - 6\lambda_1 = -8$ that is $p - 18 = -8$ which means $p = 10$.
From (7),we have $q - 5\lambda_1 = -16$ that is $q - 15 = -16$ which means $q = -1$.
Hence, the vector equation for L_1 is
$\underset{\sim}{r_1} = \begin{pmatrix} 10 \\ -2 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -6 \\ 2 \\ -5 \end{pmatrix}$.

c) We can solve $iz^2 - (i-2)z + 1 + 3i = 0$ using the quadratic formula.

$$z = \frac{i - 2 \pm \sqrt{(i-2)^2 - 4i(1+3i)}}{2i}$$

$$z = \frac{i - 2 \pm \sqrt{-1 - 4i + 4 - 4i + 12}}{2i}$$

$$z = \frac{i - 2 \pm \sqrt{15 - 8i}}{2i} \quad (1)$$

Now, let $(a+ib)^2 = 15 - 8i$

$a^2 - b^2 + 2ab\, i = 15 - 8i$

Equating reals, we get

$a^2 - b^2 = 15$ (1)

Equating imaginaries, we get

$ab = -4$ (2)

Equating moduli, we get

$a^2 + b^2 = 17$ (3)

(1) + (3) gives

$2a^2 = 32$ that is $a^2 = 16$

So, $a = \pm 4$.

Now, when $a = 4, b = -1$ and

when $a = -4, b = 1$

So, $\sqrt{15 - 8i} = \pm(4 - i)$ (4)

By substituting (4) in (1), we get

$$z = \frac{i - 2 \pm (4 - i)}{2i}$$

$$z = \frac{i - 2 + (4 - i)}{2i} \text{ or } z = \frac{i - 2 - (4 - i)}{2i}$$

$$z = \frac{2}{2i} \text{ or } z = \frac{2i - 6}{2i}$$

$$z = \frac{2}{2i} \times \frac{i}{i} \text{ or } z = \frac{2i - 6}{2i} \times \frac{i}{i}$$

$$z = -i \text{ or } z = 1 + 3i$$

d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3}{8\cos x + 10}\, dx$

Let $t = \tan\frac{x}{2}$ so $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$. This means

$dt = \frac{1}{2}(1 + \tan^2\frac{x}{2})\, dx$; that is, $dt = \frac{1}{2}(1 + t^2)dx$

$$dx = \frac{2}{1 + t^2}\, dt$$

When $x = \frac{\pi}{2}$, $t = 1$ and when $x = 0$, $t = 0$.

Also, $8\cos x + 10 = 8 \times \frac{1 - t^2}{1 + t^2} + 10$

$$= \frac{8 - 8t^2}{1 + t^2} + \frac{10 + 10t^2}{1 + t^2}$$

$$= \frac{18 + 2t^2}{1 + t^2} = \frac{2(t^2 + 9)}{1 + t^2}$$

This means $\displaystyle\frac{3}{8\cos x + 10} = \frac{3}{\frac{2(9 + t^2)}{1 + t^2}}$

$$= \frac{3(1 + t^2)}{2(9 + t^2)}$$

So $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3}{8\cos x + 10}\, dx = \int_0^1 \frac{3(1 + t^2)}{2(9 + t^2)} \times \frac{2}{1 + t^2}\, dt$

$$= \int_0^1 \frac{3}{9 + t^2}\, dt$$

$$= \left[\tan^{-1}\frac{t}{3}\right]_0^1 = \tan^{-1}\frac{1}{3}.$$

QUESTION 13

a) $z^3 + 2z^2 + pz + q = 0$ (1)

As p and q are real, all coefficients are real. Hence if $-2 + i$ is a root, then its complex conjugate $-2 - i$ must also be a root.

Now, let α be the third root.

Sum of the roots is $-2 + i - 2 - i + \alpha = -2$. This means $-4 + \alpha = -2$; that is, $\alpha = 2$.

Product of the roots is

$(-2 + i)(-2 - i)(2) = -q$ that is $-q = 2(4 - i^2)$

So, $q = -10$.

By substituting in (1) the root $z = 2$, we get

$8 + 8 + 2p - 10 = 0$ that is $p = -3$.

Alternative method

$(z + 2 - i)$ and $(z + 2 + i)$ must be factors. That means their product $(z + 2 - i)(z + 2 + i) = (z + 2)^2 - i^2$

$= z^2 + 4z + 4 + 1$

$= z^2 + 4z + 5$ is also a factor.

Now, by dividing (1) by $z^2 + 4z + 5$, we get

$$\begin{array}{r l}
 & z - 2 \\
z^2 + 4z + 5 \,\big) & z^3 + 2z^2 + pz + q \\
 & \underline{z^3 + 4z^2 + 5z} \\
 & -2z^2 + (p - 5)z + q \\
 & \underline{-2z^2 - 8z - 10} \\
 & (p - 5 + 8)z + q + 10
\end{array}$$

As the remainder should be zero then

$p - 5 + 8 = 0$ and $q + 10 = 0$

So, $p = -3$ and $q = -10$.

b) i) The radius of the sphere H is

$r^2 = (1 + 2)^2 + (5 + 1)^2 + (3 - 1)^2$.

$r^2 = 49$; that is, $r = 7$ as the radius is positive.

Hence, the equation of this sphere H is

$(x + 2)^2 + (y + 1)^2 + (z - 1)^2 = 49$ (1)

ii) By subtracting $(x + 2)^2 + (y + 1)^2 + (z - 1)^2 = 49$ and $(x + 2)^2 + (y - 3)^2 + (z - 1)^2 = 25$, we get

$(y + 1)^2 - (y - 3)^2 = 24$

$y^2 + 2y + 1 - y^2 + 6y - 9 = 24$; that is, $8y - 8 = 24$.

So, $y = 4$. This indicates that the circle lies on the plane $y = 4$ parallel to the xz plane with fixed $y = 4$.

Therefore, the equation of the circle is

$(x + 2)^2 + (z - 1)^2 = 24$ and $y = 4$.

c) i) $\cos(\alpha+\beta)+\cos(\alpha-\beta)$
$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$= 2\cos\alpha\cos\beta$.

ii) Using part (i) and let $\alpha+\beta = \frac{3x}{4}$ (1)
and $\alpha - \beta = \frac{x}{4}$ (2).
Now, by adding (1) and (2), we get
$2\alpha = x$ that is $\alpha = \frac{x}{2}$ and $\beta = \frac{3x}{4} - \frac{x}{2} = \frac{x}{4}$.
Therefore, $\cos\frac{3x}{4} + \cos\frac{x}{4} = 2\cos\frac{x}{2}\cos\frac{x}{4}$.
So, $\cos\frac{3x}{4} + 2\cos\frac{x}{4}\sin x + \cos\frac{x}{4} = 0$
becomes $2\cos\frac{x}{4}\sin x + 2\cos\frac{x}{4}\cos\frac{x}{2} = 0$
$2\cos\frac{x}{4}\left(\sin x + \cos\frac{x}{2}\right) = 0$
$2\cos\frac{x}{4}\left(2\sin\frac{x}{2}\cos\frac{x}{2} + \cos\frac{x}{2}\right) = 0$
$2\cos\frac{x}{4}\cos\frac{x}{2}\left(2\sin\frac{x}{2} + 1\right) = 0$
This means $\cos\frac{x}{4} = 0$, $\cos\frac{x}{2} = 0$ or $\sin\frac{x}{2} = -\frac{1}{2}$
$\cos\frac{x}{4} = 0 = \cos\frac{\pi}{2}$
$\frac{x}{4} = \frac{\pi}{2} + 2n\pi$ or $\frac{x}{4} = -\frac{\pi}{2} + 2n\pi$
$x = 2\pi + 8n\pi$ *or* $x = -2\pi + 8n\pi$
$n = 0, \boldsymbol{x} = 2\pi$ *is the only valid solution.*
$\cos\frac{x}{2} = 0 = \cos\frac{\pi}{2}$
$\frac{x}{2} = \frac{\pi}{2} + 2n\pi$ or $\frac{x}{2} = -\frac{\pi}{2} + 2n\pi$
$x = \pi + 4n\pi$ *or* $x = -\pi + 4n\pi$
$n = 0, x = \pi$ *is the only valid solution.*
$\sin\frac{x}{2} = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$
$\frac{x}{2} = -\frac{\pi}{6} + 2n\pi$ or $\frac{x}{2} = \pi + \frac{\pi}{6} + 2n\pi$
$x = -\frac{\pi}{3} + 4n\pi$ *or* $x = \frac{7\pi}{3} + 4n\pi$
There is no valid solution.
Hence, the only valid solutions in the domain
$0 \le x \le 2\pi$ are $x = \pi$ *or* 2π.

d) From the diagram
$\overrightarrow{PA} = \overrightarrow{PS} + \overrightarrow{SA}$ and $\overrightarrow{PC} = \overrightarrow{PS} + \overrightarrow{SC}$. This means
$\overrightarrow{PA} + \overrightarrow{PC} = \overrightarrow{PS} + \overrightarrow{SA} + \overrightarrow{PS} + \overrightarrow{SC}$
Now, as $\overrightarrow{SA}$ *and* $\overrightarrow{SC}$ are equal radii, but in opposite directions this means $\overrightarrow{SC} = -\overrightarrow{SA}$ and this indicates that $\overrightarrow{PA} + \overrightarrow{PC} = \overrightarrow{PS} + \overrightarrow{SA} + \overrightarrow{PS} - \overrightarrow{SA} = 2\overrightarrow{PS}$ (1)
Similarly, $\overrightarrow{PB} + \overrightarrow{PD} = 2\overrightarrow{PS}$ (2)
Also, $\overrightarrow{PE} = \overrightarrow{PR} + \overrightarrow{RE}$ and $\overrightarrow{PG} = \overrightarrow{PR} + \overrightarrow{RG}$
Now, as $\overrightarrow{RE}$ *and* $\overrightarrow{RG}$ are equal radii, but in opposite directions, this means $\overrightarrow{RG} = -\overrightarrow{RE}$ and this indicates that $\overrightarrow{PE} + \overrightarrow{PG} = \overrightarrow{PR} + \overrightarrow{RE} + \overrightarrow{PR} - \overrightarrow{RE} = 2\overrightarrow{PR}$ (3)
Similarly, $\overrightarrow{PF} + \overrightarrow{PH} = 2\overrightarrow{PR}$ (4)
By adding (1), (2), (3)and (4), we get
$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PG} + \overrightarrow{PH}$
$= \overrightarrow{PA} + \overrightarrow{PC} + \overrightarrow{PB} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PG} + \overrightarrow{PF} + \overrightarrow{PH}$
$= 2\overrightarrow{PS} + 2\overrightarrow{PS} + 2\overrightarrow{PR} + 2\overrightarrow{PR}$
$= 4\overrightarrow{PS} + 4\overrightarrow{PR}$ (5)
Now, as RS is a straight line and $PR = h$ and $PS = 2h$ this means that $2\overrightarrow{PR} = -\overrightarrow{PS}$.
By substituting $2\overrightarrow{PR} = -\overrightarrow{PS}$ in (5), we get
$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PG} + \overrightarrow{PH}$
$= 4\overrightarrow{PS} - 2\overrightarrow{PS} = 2\overrightarrow{PS}$

QUESTION 14

a) As the motion is centred at 2, and has amplitude 5, the velocity v will be zero at the extremities of the motion; that is, when $x = 7$ and $x = -3$.
Hence, $p + 56 - 49q = 0$; *that is* $p - 49q = -56$ (1)
and $p - 24 - 9q = 0$; *that is* $p - 9q = 24$ (2)
By Subtracting (2) from (1), we get
$-40q = -80$ So, $q = 2$.
By substituting in (2), we get
$p - 18 = 24$ So, $p = 42$.
Now, by substituting $p = 42$ *and* $q = 2$ *in*
$v^2 = p + 8x - qx^2$, we get $v^2 = 42 + 8x - 2x^2$
that is $\frac{1}{2}v^2 = 21 + 4x - x^2$
By using $a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$, we get
$a = 4 - 2x = -2(x-2)$
Therefore, $n^2 = 2$; that is, $n = \sqrt{2}$ *as* $n > 0$.
Hence, the period $T = \frac{2\pi}{n}$ *and* this means $T = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$.
OR
$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = -q\left(x - \frac{4}{q}\right)$. Since the motion is centred at 2 and comparing with $\ddot{x} = -n^2\left(x - x_0\right)$ we see that $q = 2$ and hence $n = \sqrt{2}$, etc.

b) A volume formed by rotating a function about the y axis is found using $V = \pi\int_c^d \left(f(y)\right)^2 dy$. So we will need to rewrite the functions with x as the subject.

$y = 2\sin^{-1} x$ and $y = 2\sin^{-1}\frac{x}{2}$

$\frac{y}{2} = \sin^{-1} x$ and $\frac{y}{2} = \sin^{-1}\frac{x}{2}$

$x = \sin\frac{y}{2}$ and $\frac{x}{2} = \sin\frac{y}{2}$

$x^2 = \sin^2\frac{y}{2}$ and $x^2 = 4\sin^2\frac{y}{2}$

$V_1 = \pi\int_0^p 4\sin^2\frac{y}{2} - \sin^2\frac{y}{2}\ dy$

$V_1 = 3\pi\int_0^p \sin^2\frac{y}{2}\ dy$

$V_1 = 3\pi\int_0^p \frac{1}{2}(1-\cos y)\ dy$

$V_1 = \frac{3\pi}{2}[y - \sin y]_0^p$

$V_1 = \frac{3\pi}{2}(p - \sin p)$

Similarly, $V_2 = \frac{3\pi}{2}[\ y - \sin y\ ]_p^\pi$

$V_2 = \frac{3\pi}{2}[(\pi - 0\ ) - (p - \sin p)]$

$V_2 = \frac{3\pi}{2}(\pi - p + \sin p)$

Now, as $V_1 = 2V_2\ then$

$\frac{3\pi}{2}(\ p - \sin p) = 2\times\frac{3\pi}{2}[\ \pi - p + \sin p\ ]$

$p - \sin p = 2\pi - 2p + 2\sin p$

$3p - 3\sin p = 2\pi$

$3\sin p = 3p - 2\pi$

$\sin p = p - \frac{2\pi}{3}$ but $\sin p\ \approx\ p^2 - 2p - 1.06612$, and this means $p^2 - 2p - 1.06612 = p - \frac{2\pi}{3}$.

That is $3p^2 - 6p - 3.19836 = 3p - 2\pi$

$3p^2 - 9p + 2\pi - 3.19836 = 0$

Using the quadratic formula, we get

$p = 2.60532\ or\ 0.3946.$

But $0.3946\ ... < \frac{\pi}{2}$, which means that

$p = 2.60532\ (\ 5\ d.p.)$ is the only valid solution.

c) Assume that $\sqrt[3]{3}$ is rational. Let $\sqrt[3]{3} = \frac{p}{q}$, where p and q are positive integers with no common factor.

By cubing both sides, we get $3 = \frac{p^3}{q^3}$, that is $3q^3 = p^3$. (1)

This shows p^3 is a multiple of 3 and this indicates that p is a multiple of 3 appealing to Euclid's Lemma. Therefore, p could be written as $3r$.

By substituting $p = 3r$ in (1), we get

$3q^3 = 27r^3$ that is $q^3 = 9r^3 = 3\times 3r^3$

This means that q^3 can be divided 3, and so appealing to Euclid's Lemma once more, we conclude q can be divided by 3. From the above, we can say that 3 is a common factor of p and q, but from our original assertion, p and q have no common factor, other than 1, and so there is a contradiction.

Hence ,$\sqrt[3]{3}$ is not rational; so it must be irrational.

d) $12^n = (5+7)^n$

$= \binom{n}{0}5^n + \binom{n}{1}5^{n-1}\times 7 + \binom{n}{2}5^{n-2}\times 7^2 +$

$... + \binom{n}{n-1}\times 5\times 7^{n-1} + \binom{n}{n}7^n$

As $\binom{n}{0} = \binom{n}{n} = 1$ then

$(5+7)^n = 5^n + \binom{n}{1}5^{n-1}\times 7 + \binom{n}{2}5^{n-2}\times 7^2 +$

$... + \binom{n}{n-1}\times 5\times 7^{n-1} + 7^n$

Since $\binom{n}{1}5^{n-1}\times 7 + \binom{n}{2}5^{n-2}\times 7^2 +$

$... + \binom{n}{n-1}\times 5\times 7^{n-1}$ is a sum of positive terms then $(5+7)^n = 5^n + 7^n + Sum\ of\ positive\ terms$. This indicates that $(5+7)^n > 5^n + 7^n$; that is,

$12^n > 5^n + 7^n$ for all integers $n \geq 2$.

e) In order to prove $P \Leftrightarrow Q\ is\ true$ we need to prove that $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true.

So, let us now prove $P \Rightarrow Q$; $that\ is$,

"$n + 21$ is a positive even integer if $n + 6$ is a positive odd integer."

Let $n + 6$ be a positive odd integer. This means

$n + 6 = 2r + 1$, where r is a positive integer. Then

$n = 2r - 5$ with $r > 2.5$. (Why?)and so:

$n + 21 = 2r - 5 + 21$

$= 2r + 16 = 2\ (r + 8)$, which is even.

So, $n + 21$ is a positive even integer if $n + 6$ is odd.

So, let us now prove $Q \Rightarrow P$; $that\ is$, if $n + 21$ is a positive even integer, then $n + 6$ is a positive odd integer.

Let $n + 21$ be a positive even integer. Then $n + 21 = 2p$, where p is an integer and this means $n = 2p - 21$ with $p > 10.5$. (Why?)

So, $n + 6\ =\ 2p - 21 + 6$

$= 2p - 15$

$= 2p - 2\times 8 + 1$

$= 2\ (p - 8) + 1$, which will be odd for all integer values of p and certainly for $p > 10.5$. So If $n + 21$ is an even positive integer, then $n + 6$ is a positive odd integer.

Hence, the logical equivalence is true.

QUESTION 15

a) From the statement $3^x > x + lnx$

When $x = 1$, $3^1 > 1 + \ln 1$

When $x = 2$, $3^2 > 2 + \ln 2$

When $x = 3$, $3^3 > 3 + \ln 3$

$\vdots$

When $x = n$, $3^n > n + \ln n$

By adding the terms on left hand side, we get

$3^1 + 3^2 \ldots. + 3^n = \frac{3(3^n - 1)}{3-1} = \frac{3^{n+1} - 3}{2}$

Also, by adding on the right hand side, we get

$1 + 2 + 3 + .. + n = \frac{n}{2}(n+1) = \frac{1}{2}(n^2 + n)$

and $\ln 1 + \ln 2 \ldots + \ln n = \ln(1 \times 2 \times 3 .. \times n)$

$= \ln n!$

Hence, by adding all of the statements for $x = 1$ to $x = n$

$$\frac{3^{n+1} - 3}{2} > \frac{1}{2}(n^2 + n) + \ln n!$$

So $3^{n+1} - 3 > n^2 + n + 2\ln n!$

Hence, $3^{n+1} > n^2 + n + 3 + 2\ln n!$

b) i) When $t = 1$, $\underset{\sim}{r_1} = \begin{pmatrix} ln\, a - 2 \\ ln\, a - 1 \\ 5 \end{pmatrix}$ & $\underset{\sim}{r_2} = \begin{pmatrix} ln\, a \\ 2\ ln\, a + 1 \\ 3\, ln\, a + 3 \end{pmatrix}$

So, after one second the distance between P_1 and P_2 *will be*

$d = \sqrt{2^2 + (ln\, a + 2)^2 + (3 lna - 2)^2}$

The distance between the particles when $t = 1$ is 6 m, so by squaring both sides, we get

$36 = 4 + ln^2 a + 4\ ln\, a + 4 + 9\, ln^2 a - 12\ ln\, a + 4$

$36 = 10\, ln^2 a - 8\, ln\, a + 12$

$10\, ln^2 a - 8\, ln\, a - 24 = 0$

$5\, ln^2 a - 4\ ln\, a - 12 = 0$

$(5\ ln\, a + 6)(\, ln\, a - 2) = 0$

So, $ln\, a = -\frac{6}{5}$ *or* $ln\, a = 2$

As $a > 1$, *then the only valid answer is* $ln\, a > 0$.

So, $ln\, a = 2$.

b) ii)

Now, by substituting $ln\, a = 2$ into the equations

$\underset{\sim}{r_1} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix}$

When $t = 2$, $\underset{\sim}{r_1} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ *and* $\underset{\sim}{r_2} = \begin{pmatrix} 6 \\ 11 \\ 15 \end{pmatrix}$

The distance between P_1 and P_2 after 2 seconds will be

$d = \sqrt{(6-2)^2 + (11-3)^2 + (15-7)^2}$

$d = 12$ m

c) i) $m\frac{dv}{dt} = -mg - m\sqrt{\frac{v}{g}}$

$\frac{dv}{dt} = -g - \frac{\sqrt{v}}{\sqrt{g}}$

$\frac{dv}{dt} = -\left(g + \frac{\sqrt{gv}}{g}\right)$

$\frac{dv}{dt} = -\left(\frac{g^2 + \sqrt{gv}}{g}\right)$

This means $\frac{dt}{dv} = -\frac{g}{g^2 + \sqrt{gv}}$.

Integrating both sides, we get

$\int dt = \int -\frac{g}{g^2 + \sqrt{gv}}\, dv$

$t = -\int \frac{g}{g^2 + \sqrt{gv}}\, dv \quad (1)$

c) ii)

Let $u = \sqrt{gv}$. This means

$\frac{du}{dv} = \frac{g}{2\sqrt{gv}}$ that is

$du = \frac{g}{2\sqrt{gv}}\, dv$. This means

$du = \frac{g}{2u}\, dv$ so, $2u\, du = g\, dv$

From (1), we have

$t = -\int \frac{g}{g^2 + \sqrt{gv}}\, dv$; that is,

$t = -\int \frac{2u}{g^2 + u}\, du$

$t = -2\int \frac{u + g^2 - g^2}{g^2 + u}\, du$

$t = -2\int \frac{g^2 + u}{g^2 + u} - \frac{g^2}{g^2 + u}\, du$

$t = -2\int 1 - \frac{g^2}{g^2 + u}\, du$

$t = -2\left(u - g^2 \ln\left(g^2 + u\right)\right) + c$

$t = -2u + 2g^2 \ln(g^2 + u) + c$

$t = -2\sqrt{gv} + 2g^2 \ln\left(g^2 + \sqrt{gv}\right) + c$

When $t = 0, v = v_o$ and this means

$0 = -2\sqrt{gv_o} + 2g^2 \ln\left(g^2 + \sqrt{gv_o}\right) + c$

$c = 2\sqrt{gv_o} - 2g^2 \ln\left(g^2 + \sqrt{gv_o}\right)$

$t = -2\sqrt{gv} + 2g^2 \ln\left(g^2 + \sqrt{gv}\right) + 2\sqrt{gv_o}$
$\quad - 2g^2 \ln\left(g^2 + \sqrt{gv_o}\right)$

$t = 2\left(\sqrt{gv_o} - \sqrt{gv}\right) + 2g^2 \ln\left(\frac{g^2 + \sqrt{gv}}{g^2 + \sqrt{gv_o}}\right)$

ii) When $v = \frac{v_o}{9}$,

$$t = 2\left(\sqrt{gv_o} - \sqrt{g\frac{v_o}{9}}\right) + 2g^2 \ln\left(\frac{g^2 + \sqrt{g\frac{v_o}{9}}}{g^2 + \sqrt{gv_o}}\right)$$

$$t = 2\left(\sqrt{gv_o} - \frac{1}{3}\sqrt{gv_o}\right) + 2g^2 \ln\left(\frac{g^2 + \frac{1}{3}\sqrt{gv_o}}{g^2 + \sqrt{gv_o}}\right)$$

$$t = \frac{4}{3}\sqrt{gv_o} + 2g^2 \ln\left(\frac{3g^2 + \sqrt{gv_o}}{3g^2 + 3\sqrt{gv_o}}\right)$$

When $v = \frac{v_o}{81}$,

$$t = 2\left(\sqrt{gv_o} - \sqrt{g\frac{v_o}{81}}\right) + 2g^2 \ln\left(\frac{g^2 + \sqrt{g\frac{v_o}{81}}}{g^2 + \sqrt{gv_o}}\right)$$

$$t = 2\left(\sqrt{gv_o} - \frac{1}{9}\sqrt{gv_o}\right) + 2g^2 \ln\left(\frac{g^2 + \frac{1}{9}\sqrt{gv_o}}{g^2 + \sqrt{gv_o}}\right)$$

$$t = \frac{16}{9}\sqrt{gv_o} + 2g^2 \ln\left(\frac{9g^2 + \sqrt{gv_o}}{9g^2 + 9\sqrt{gv_o}}\right)$$

Hence, the time taken by the particle to drop its velocity from $\frac{v_o}{9}$ to $\frac{v_o}{81}$ is

$$t = \frac{16}{9}\sqrt{gv_o} + 2g^2 \ln\left(\frac{9g^2 + \sqrt{gv_o}}{9g^2 + 9\sqrt{gv_o}}\right) - \frac{4}{3}\sqrt{gv_o}$$
$$- 2g^2 \ln\left(\frac{3g^2 + \sqrt{gv_o}}{3g^2 + 3\sqrt{gv_o}}\right)$$

$$t = \frac{4}{9}\sqrt{gv_o} + 2g^2 \ln\left(\frac{9g^2 + \sqrt{gv_o}}{9g^2 + 9\sqrt{gv_o}}\right) \times \left(\frac{3g^2 + 3\sqrt{gv_o}}{3g^2 + \sqrt{gv_o}}\right)$$

$$t = \frac{4}{9}\sqrt{gv_o} + 2g^2 \ln\left(\frac{9g^2 + \sqrt{gv_o}}{9g^2 + 3\sqrt{gv_o}}\right).$$

QUESTION 16

a) The most effective way to nominate an even number is to designate it as $2n$, where n is an integer.
This means the three consecutive even integers could be $2n, 2n+2$ and $2n+4$, where n is a positive integer. Let the sum of the squares of these three consecutive even integers be denoted by S. Our goal is to show:

$S = 12M + 8$ where $M \in \mathbb{N}$

$$\begin{aligned} S &= (2n)^2 + (2n+2)^2 + (2n+4)^2 \\ &= 4n^2 + 4n^2 + 8n + 4 + 4n^2 + 16n + 16 \\ &= 12n^2 + 24n + 20 \\ &= 12n^2 + 24n + 12 + 8 \\ &= 12(n^2 + 2n + 1) + 8 \\ &= 12(n+1)^2 + 8 \\ &= 12M + 8; M = (n+1)^2 \in \mathbb{N}. \end{aligned}$$

From the result above we can see that the sum of any three consecutive even integers can be divided by 12 and leave a remainder of 8

b) i) a = $v\frac{dv}{dx} = \frac{-1}{2x^2}$

Integrating both sides, we get

$$\int v\, dv = \int \frac{-1}{2x^2} dx$$

$$\frac{1}{2}v^2 = -\frac{1}{2}\int x^{-2} dx$$

$$\frac{1}{2}v^2 = \frac{1}{2x} + c$$

When $t = 0$, $x = \frac{1}{2}$ and $v = 1$.

$\frac{1}{2} = 1 + c$ that is $c = -\frac{1}{2}$.

$$\frac{1}{2}v^2 = \frac{1}{2x} - \frac{1}{2}$$

$v^2 = \frac{1}{x} - 1$; that is, $v^2 = \frac{1-x}{x}$

So, $v = \pm\sqrt{\frac{1-x}{x}}$ As $v = 1$ when $x = \frac{1}{2}$

then $v = \sqrt{\frac{1-x}{x}}$ is valid; that is, $\frac{dx}{dt} = \sqrt{\frac{1-x}{x}}$.

ii) $\frac{dx}{dt} = \sqrt{\frac{1-x}{x}}$; *that is*, $\frac{dt}{dx} = \sqrt{\frac{x}{1-x}}$

Now, the time T taken for the particle to travel from $x = \frac{1}{2}$ to $x = 1$ is $T = \int_{\frac{1}{2}}^{1} \sqrt{\frac{x}{1-x}}\, dx.$

Let $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = \frac{1}{2}$, $\sin^2\theta = \frac{1}{2}$; that is,

$\sin\theta = \frac{1}{\sqrt{2}}$ and hence, $\theta = \frac{\pi}{4}$.

When $x = 1$, $\sin^2\theta = 1$; that is,

$\sin\theta = 1$ and hence, $\theta = \frac{\pi}{2}$.

Strictly $\theta = \sin^{-1}\left(\sqrt{x}\right)$ or $\theta = \sin^{-1}\left(-\sqrt{x}\right)$; that is, two substitutions are embedded in $x = \sin^2\theta$.

$$T = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{\sin^2\theta}{1 - \sin^2\theta}} \times 2\sin\theta\cos\theta\, d\theta$$

$$T = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \times 2\sin\theta\cos\theta\, d\theta$$

$T = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\, d\theta$ since

$\sin\theta \,\&\, \cos\theta \geq 0$ on $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

$$T = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin^2\theta\, d\theta$$

$$T = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2\theta)\, d\theta$$

$$T = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$T = \frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$T = \left(\frac{\pi}{4} + \frac{1}{2}\right)$ seconds.

c) i) $I_n = \int_0^1 \frac{x^3}{(2-x^2)^n}\, dx$

Using integration by parts and letting

$u = (2-x^2)^{-n}$ and $dv = x^3\, dx$

$du = -n(2-x^2)^{-n-1} \times -2x\, dx \quad v = \frac{x^4}{4}$

$$du = \frac{2nx}{(2-x^2)^{n+1}}\, dx$$

So $I_n = \left[\frac{x^4}{4}(2-x^2)^{-n}\right]_0^1 - \int_0^1 \frac{x^4}{4} \times \frac{2nx}{(2-x^2)^{n+1}}\, dx$

$$I_n = \frac{1}{4} - \frac{n}{2}\int_0^1 \frac{x^5}{(2-x^2)^{n+1}}\, dx$$

Now, using the given identity, we get

$$\frac{x^5}{(2-x^2)^{n+1}} = \frac{Ax^3}{(2-x^2)^n} + \frac{Bx^3}{(2-x^2)^{n+1}}$$

Multiplying by $(2-x^2)^{n+1}$, we get

$x^5 = Ax^3(2-x^2) + Bx^3$

$x^5 = 2Ax^3 - Ax^5 + Bx^3$

$x^5 = -Ax^5 + (2A+B)x^3$

So $-A = 1$ and $2A + B = 0$

$A = -1$ $\quad 2 \times -1 + B = 0$ that is $B = 2$.

$$I_n = \frac{1}{4} - \frac{n}{2}\left[\int_0^1 \frac{-x^3}{(2-x^2)^n} + \frac{2x^3}{(2-x^2)^{n+1}}\, dx\right]$$

$$I_n = \frac{1}{4} + \frac{n}{2}\int_0^1 \frac{x^3}{(2-x^2)^n}\, dx - n\int_0^1 \frac{x^3}{(2-x^2)^{n+1}}\, dx$$

$$I_n = \frac{1}{4} + \frac{n}{2} I_n - n I_{n+1}$$

$$n I_{n+1} = \frac{1}{4} + \frac{n}{2} I_n - I_n$$

$$n I_{n+1} = \frac{1}{4} + \frac{n-2}{2} I_n$$

$$I_{n+1} = \frac{1}{4n} + \frac{n-2}{2n} I_n$$

ii) $\int_0^1 \frac{x^3}{(2-x^2)^2}\, dx = I_2$

From part i), $I_2 = \frac{1}{4} + \frac{-1}{2} I_1$

$$I_1 = \int_0^1 \frac{x^3}{2-x^2}\, dx$$

$$\begin{array}{r|l} & -x \text{ remainder } 2x \\ \hline -x^2+2 & x^3 + 0x^2 + 0x \\ & x^3 + 0x^2 - 2x \\ \hline & 2x \end{array}$$

So, $\frac{x^3}{2-x^2} = -x + \frac{2x}{2-x^2}$ and this means

$$I_1 = \int_0^1 \frac{x^3}{2-x^2}\, dx$$

$$I_1 = \int_0^1 -x + \frac{2x}{2-x^2}\, dx$$

$$I_1 = \left[-\frac{x^2}{2} - \ln(2-x^2)\right]_0^1$$

$$I_1 = -\frac{1}{2} - \ln 1 - (-\ln 2)$$

$$I_1 = -\frac{1}{2} + \ln 2$$

By substituting $I_1 = -\frac{1}{2} + \ln 2$ in $I_2 = \frac{1}{4} + \frac{-1}{2} I_1$, we get

$$I_2 = \frac{1}{4} + \frac{-1}{2}\left(-\frac{1}{2} + \ln 2\right)$$

$$I_2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{2}\ln 2$$

$$I_2 = \frac{1}{2} - \frac{1}{2}\ln 2$$

Multiple Choice Summary Answers

1. D	**6. B**
2. A	**7. A**
3. C	**8. B**
4. D	**9. D**
5. C	**10. A**

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